GROUND STATE MASS CONCENTRATION IN THE L^2-CRITICAL NONLINEAR SCHRÖDINGER EQUATION BELOW H^1

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ABSTRACT. We consider finite time blowup solutions of the L^2-critical cubic focusing nonlinear Schrödinger equation on $\mathbb{R}^2$. Such functions, when in H^1, are known to concentrate a fixed L^2-mass (the mass of the ground state) at the point of blowup. Blowup solutions from initial data that is only in L^2 are known to concentrate at least a small amount of mass. In this paper we consider the intermediate case of blowup solutions from initial data in H^s, with $1 > s > s_Q$, where $s_Q \leq \frac{1}{5} + \frac{1}{5}\sqrt{11}$. Our main result is that such solutions, when radially symmetric, concentrate at least the mass of the ground state at the origin at blowup time.

1. INTRODUCTION

Special interest has recently been devoted to the existence and long-time behavior of solutions with low regularity to nonlinear Schrödinger equations. These questions were mainly investigated for defocusing[1] equations with a global-in-time a priori H^1 upper bound [1] [6] [21] [7]. In this article, we are interested in a detailed description of rough solutions in a situation where the H^1-norm of certain smooth solutions blows up in a finite time.

We consider the initial value problem for the two-dimensional, cubic, focusing nonlinear Schrödinger (NLS) equation:

$$(1.1) \qquad \begin{cases} iu_t + \Delta u + |u|^2 u = 0, \\ \quad u(0,x) = u_0(x), \qquad x \in \mathbb{R}^2, \end{cases}$$

which is L^2-critical. This refers to the property that both the equation and the L^2-norm of the solution are invariant under the scaling transformation

[1]Low regularity global well-posedness results have also been obtained for other Hamiltonian evolution equations. Orbital instability properties of solitons subject to rough perturbations for focusing NLS equations have been studied as well [8], [9].

$u(t,x) \rightarrow u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$. This problem is locally well-posed[2] for initial data in H^s with $s \geq 0$ [4]. We recall that the following quantities, if finite, are conserved:

$$Mass = M[u(t)] = \|u(t)\|_{L^2}^2,$$

$$Energy = E[u(t)] = \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 - \frac{1}{4}\|u(t)\|_{L^4}^4.$$

We will frequently refer to $\frac{1}{2}\|\nabla u(t)\|_{L^2}^2$ as the *kinetic energy* of the solution, and $E[u(t)]$ as the *total energy*.

It is known that there exist explicit finite time blowup solutions to (1.1); for sufficiently smooth and decaying initial data, the virial identity provides a sufficient condition guaranteeing that finite time blowup occurs. For a solution which blows up in finite time, let $[0, T^*)$ be the maximal (forward) time interval of existence.

A specific property of critical collapse is the phenomenon of mass concentration, often referred to in the physical literature as *strong collapse* [24] (see also [17] for a review): H^1-solutions concentrate a finite amount of mass in a neighborhood of the focus of width slightly larger than $(T^* - t)^{1/2}$. Heuristic arguments suggest that this phenomenon does not occur in supercritical nonlinear Schrödinger blowup. For H^1-solutions of (1.1), there is a precise lower bound on the amount of concentrated mass, namely the mass of the ground state Q [20], [15], where Q is the unique positive solution (up to translations) of

(1.2) $$\Delta w - w + |w|^2 w = 0.$$

In addition to the scaling properties of the NLS equation, the main ingredients in the proof that H^1 blowup solutions concentrate at least the mass of the ground state are: (i) the conservation of the energy, (ii) a precise Gagliardo-Nirenberg inequality [22] which implies that nonzero H^1-functions of non-positive energy have at least ground state mass.

The purpose of this work is to address the phenomenon of mass concentration in the spaces H^s, $s < 1$, where the conservation of energy cannot be used. In the setting of merely L^2 initial data, if global well-posedness fails to hold for (1.1) (*i.e.* $T^* < \infty$), then a nontrivial parabolic concentration of L^2-mass occurs [1] as $t \uparrow T^*$:

(1.3) $$\limsup_{t\uparrow T^*} \sup_{\substack{\text{cubes } I \subset \mathbb{R}^2 \\ \text{side}(I) < (T^* - t)^{\frac{1}{2}}}} \left(\int_I |u(t,x)|^2 dx \right)^{\frac{1}{2}} \geq \eta(\|u_0\|_{L^2}) > 0.$$

Unlike the H^1-case, there is no explicit quantification on the lower bound $\eta(\|u_0\|_{L^2})$.

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[2]For $s = 0$, the size of the interval of existence depends upon the initial profile u_0 [3]; for $s > 0$, the H^s norm of the data determines the size of the existence interval.

A natural question[3], highlighted in [14], is to determine whether tiny L^2-mass concentrations can occur when $u_0 \in L^2$. The conjectured answer is no. Solutions of (1.1) with a finite maximal (forward) existence interval are expected to concentrate at least the L^2-mass of the ground state. Our main result corroborates this expectation, at least for H^s-solutions with s just below 1.

Theorem 1.1. *There exists $s_Q \leq \frac{1}{5} + \frac{1}{5}\sqrt{11}$ such that the following is true for any $s > s_Q$. Suppose $H^s \ni u_0 \longmapsto u(t)$ solves (1.1) on the maximal (forward) time interval $[0, T^*)$, with $T^* < \infty$. Moreover, assume that u_0 is radially symmetric.[4] Then for any positive $\gamma(z) \uparrow \infty$ arbitrarily slowly as $z \downarrow 0$ we have*

$$(1.4) \qquad \limsup_{t \uparrow T^*} \|u\|_{L^2_{\{|x| < (T^*-t)^{s/2}\gamma(T^*-t)\ \}}} \geq \|Q\|_{L^2}.$$

The proof consists of an imitation of the H^1 argument (as presented in [2]) with the energy, which is infinite in the H^s-setting, replaced by a modified energy introduced in [6]. The idea is to apply to the H^s-solution a smoothing operator to make it H^1 and define the usual energy of this new object. The crucial point here is to prove that the modified total energy grows more slowly than the modified kinetic energy. In Section 2, we prove Theorem 1.1 assuming Proposition 2.1 and Corollary 3.6 stated below. Proposition 2.1 contains a key upper bound for the modified energy in terms of the H^s-norm of the solution. Its proof, which relies upon the local-in-time theory developed in Section 3, is postponed to Section 4.

Remark 1.2. *As mentioned above, the expectation is that parabolic concentration of at least the ground state mass holds true even for L^2 initial data. However, the analysis showing that the modified total energy grows more slowly than the modified kinetic energy (quantified by the statement that $p(s) < 2$ in Proposition 2.1), requires taking s close to 1. Also, note that the concentration width $(T^* - t)^{s/2+}$ obtained in (1.4) is much larger than $(T^* - t)^{1/2+}$ within which ground state mass concentration is expected to occur.*

Remark 1.3. *The two-dimensional nature of the analysis only appears directly in the use of the Sobolev inequality in the proof of Lemma 3.9. In higher dimensions the power of the nonlinearity of the L^2-critical problem is non-integer, which would make the multilinear analysis of the modified energy more technical.*

Remark 1.4. *Our methods only enable us to control the modified energy at time t by the supremum up to time t of the modified kinetic energy. This is*

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[3]S. Keraani has announced that the lower bound in (1.3) may be taken to be a fixed constant δ_0 independent of the initial size in L^2.

[4]The non-radial case is amenable to treatment by employing the methods of compensated compactness as used in [23], [16] (see also [2]).

the source of the lim sup *in Theorem 1.1, which does not appear in the H^1 result in* [20]. *Because there is no known upper bound for the rate of blowup of the H^s norm, we are unable to control the relationship between times at which the blowup is occurring maximally and those where it is occurring minimally. Any such result would, in addition to its inherent interest, allow us to strengthen our result to one with a* lim inf. *For further discussion of the monotonicity of the blowup, but in the context of the critical generalized KdV equation, see pp. 621–623 of* [13].

Remark 1.5. *We use C to denote constants which do not depend on the crucial parameters and variables. Such constants will not depend on the high-frequency cut-off parameter (denoted N) or the time t, but may depend on s, T^*, or $\|u_0\|_{H^s}$.*

In the proof of Theorem 1.1, nowhere do we make the hypothesis that $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$. Thus, since solutions conserve mass and the concentration is shown to be in excess of the ground state mass, we prove the following corollary about the global well-posedness of (1.1).

Corollary 1.6. *There exists $s_Q \leq \frac{1}{5} + \frac{1}{5}\sqrt{11}$ such that, if $u_0 \in H^s$, $s > s_Q$ is radially symmetric and $\|u_0\|_{L^2} < \|Q\|_{L^2}$ then the initial value problem (1.1) is globally well-posed.*

Remark 1.7. [5] *A statement stronger than Corollary 1.6 may be inferred from earlier work. The analysis in* [6] *of the defocusing analog of* (1.1) *relies upon the local well-posedness theory and two other inputs: the almost conservation of the modified energy (Proposition 3.1 of* [6]*) and the obvious fact that, in the defocusing case, the modified total energy controls the modified kinetic energy. The almost conservation property depends upon the local-in-time space-time boundedness properties of the solution and also holds in the focusing case. Since the smoothing operator I_N (see* (2.1) *below) shrinks the L^2 size of functions, it can be shown that the modified total energy does indeed control the modified kinetic energy provided that the initial data satisfies the smallness condition $\|u_0\|_{L^2} < \|Q\|_{L^2}$. Thus, Corollary 1.6 actually holds without the radial symmetry restriction and for $s > \frac{4}{7}$.*

2. Mass Concentration

We wish to find a replacement for energy conservation. To do so, we define the smoothing operators $I_N : H^s \to H^1$ used in [6]:

$$(2.1) \qquad \widehat{I_N u}(\xi) = m(\xi)\hat{u}(\xi)$$

[5]This remark emerged in correspondence among the authors of [6].

where

$$(2.2) \qquad m(\xi) = \begin{cases} 1, & |\xi| \leq N \\ (\frac{|\xi|}{N})^{s-1}, & |\xi| > 3N \end{cases}$$

with $m(\xi)$ smooth, radial, and monotone in $|\xi|$. Note that we will sometimes drop the subscript N when that will not lead to confusion. The following properties of I_N are easily verified:

$$(2.3) \qquad \begin{aligned} \|I_N u\|_{L^2} &\leq \|u\|_{L^2} \\ \|u\|_{H^s} &\leq \|I_N \langle D \rangle u\|_{L^2} \leq N^{1-s} \|u\|_{H^s}, \end{aligned}$$

where $\langle D \rangle$ is the multiplier operator with symbol $(1 + |\xi|^2)^{1/2}$. We define the *blowup parameters* associated to the H^s-norm of the solution

$$(2.4) \qquad \lambda(t) = \|u(t)\|_{H^s}$$

and

$$(2.5) \qquad \Lambda(t) = \sup_{0 \leq \tau \leq t} \lambda(\tau).$$

We also define the *modified blowup parameters*

$$(2.6) \qquad \sigma(t) = \|I_N \langle D \rangle u(t)\|_{L^2}$$

and

$$(2.7) \qquad \Sigma(t) = \sup_{0 \leq \tau \leq t} \sigma(\tau).$$

For finite time blowup solutions u, the blowup parameters $\sigma(t)$ and $\lambda(t)$ tend to ∞ as $t \to T^*$, and can be compared using (2.3).

We will exploit the freedom to choose N very large. Indeed, for fixed $T < T^*$, we will show that $N = N(T)$ may be chosen so large that the modified energy $E[I_N u(T)]$ is much smaller than the modified kinetic energy $\|I_N \nabla u(T)\|_{L^2}^2$.

Proposition 2.1. *There exists $s_Q \leq \frac{1}{5} + \frac{1}{5}\sqrt{11}$ such that for all $s > s_Q$ there exists $p(s) < 2$ with the following holding true: If $H^s \ni u_0 \longmapsto u(t)$ solves (1.1) on a maximal (forward) finite existence interval $[0, T^*)$ then for all $T < T^*$ there exists $N = N(T)$ such that*

$$(2.8) \qquad |E[I_{N(T)} u(T)]| \leq C_0 (\Lambda(T))^{p(s)}$$

with $C_0 = C_0(s, T^*, \|u_0\|_{H^s})$. Moreover, $N(T) = C(\Lambda(T))^{\frac{p(s)}{2(1-s)}}$.

Remark 2.2. *For $s \geq 1$ we set $p = 0$. In this case, Proposition 2.1 is reduced to the statement that the energy remains bounded, which is true since it is conserved. Note also that we have chosen $N = N(\Lambda)$ so the time dependence of N only enters through that of the blowup parameter $\Lambda = \Lambda(t)$.*

Proposition 2.1 gives a control on the growth of the modified energy as t approaches T^*. During the finite time blowup evolution, the modified kinetic energy explodes to infinity. The above Proposition shows that the total modified energy grows more slowly than its kinetic component. It is the key element in the proof of Theorem 1.1, requiring somewhat delicate harmonic analysis estimates to prove an almost conservation property of the modified energy $E[I_N u]$ in terms of space-time control given by local existence theory. For the sake of a clear presentation, we start by proving Theorem 1.1 assuming Proposition 2.1 and a lower bound on the rate of blowup of $\|I_N \langle D \rangle u\|_{L^2}$ expressed in Corollary 3.6.

Proof of Theorem 1.1. It is carried out in four steps.
   a. Rescaling and weak convergence.
Let $\{t_n\}_{n=1}^\infty$ be a sequence such that $t_n \uparrow T^*$ and for each t_n

$$\|u(t_n)\|_{H^s} = \Lambda(t_n).$$

We call such a sequence *maximizing* and denote $u_n = u(t_n)$. Define

$$I_N u_n = I_{N(t_n)} u(t_n)$$

with $N(t_n)$ taken[6] as in the Proposition 2.1. We rescale these as follows:

$$(2.9) \qquad\qquad v_n(y) = \frac{1}{\sigma_n} I_N u_n \left(\frac{y}{\sigma_n} \right)$$

where

$$\sigma_n = \|I_N \langle D \rangle u_n\|_{L^2} = \sigma(t_n).$$

We have, from (2.3), conservation of the L^2 norm and $\Lambda_n \to \infty$, that

$$(2.10) \qquad\qquad \Lambda(t_n) \le \sigma_n$$

along a maximizing sequence so $\sigma_n \to \infty$ as $n \to \infty$. The lower bound of σ_n by $\Lambda(t_n)$ does not necessarily hold for arbitrary sequences, but does so along maximizing sequences.

The rescaling (2.9) leaves L^2-norms unchanged. Since $\|I_N \cdot \|_{L^2} \le \|\cdot\|_{L^2}$ for all N and since mass is conserved, we have $\|v_n\|_{L^2} \le \|u_0\|_{L^2}$ uniformly in n. By our choice of σ_n, we have $\|\nabla v_n\|_{L^2} \le 1$ for all n. In fact, by construction, we have

$$(2.11) \qquad\qquad \lim_{n \to \infty} \|\nabla v_n\|_{L^2} = 1.$$

Thus, $\{v_n\}$ is a bounded sequence in H^1 and has a weakly convergent subsequence, which we also call $\{v_n\}$. There exists an asymptotic object $v \in H^1$ such that

$$v_n \rightharpoonup v$$

in H^1.
   b. Compactness and energy of the rescaled asymptotic object.
   Since u is assumed to be radially symmetric, so are the v_n, and we can apply the following Lemma from [18].

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[6]The reason for this choice is made clear at equation (4.2).

Lemma 2.3. *(Radial compactness lemma) If $\{f_n\} \subset H^1(\mathbb{R}^2)$ is a bounded sequence of radially symmetric functions, then there exists a subsequence (also denoted $\{f_n\}$) and a function $f \in H^1(\mathbb{R}^2)$ such that for all $2 < q < \infty$,*

$$f_n \to f$$

in L^q.

Thus,

$$v_n \to v$$

in L^4. This strong L^4-convergence is important due to the appearance of the L^4 norm in the energy, which we now examine. The only usage of the radial symmetry assumption in the argument is to obtain compactness of $\{v_n\}$ in L^4.

We have, from Proposition 2.1 and (2.10),

$$\begin{aligned}
|E[v_n]| &= \sigma_n^{-2}|E[I_N u_n]| \\
&\leq C\sigma_n^{-2}\Lambda^{p(s)}(t_n) \\
&\leq C\Lambda^{p(s)-2}(t_n).
\end{aligned}$$

Since $s > s_Q$, we have $p(s) < 2$, so

$$|E[v_n]| \to 0$$

as $n \to \infty$.

The fact that the energy of the functions v_n goes to zero is useful in two important ways. First, by (2.11) and the strong L^4-convergence,

$$\begin{aligned}
(2.12) \qquad 0 &= \lim_{n\to\infty} |E[v_n]| \\
&= \lim_{n\to\infty} \left| \frac{1}{2}\|\nabla v_n\|_{L^2}^2 - \frac{1}{4}\|v_n\|_{L^4}^4 \right| \\
&= \left| \frac{1}{2} - \frac{1}{4}\|v\|_{L^4}^4 \right|.
\end{aligned}$$

This ensures that $v \neq 0$. Second, the L^2 norm is a lower semi-continuous function for weakly convergent sequences, so

$$0 = \lim_{n\to\infty} E[v_n] \geq E[v].$$

c. Non-positive energy implies at least ground state mass.

The fact that nonzero functions of nonpositive energy have at least the mass of the ground state is a consequence of the Gagliardo–Nirenberg inequality:

$$(2.13) \qquad \|w\|_{L^4}^4 \ \leq \ C_{opt}\|w\|_{L^2}^2\|\nabla w\|_{L^2}^2.$$

The optimal constant, obtained by minimizing the functional

$$(2.14) \qquad J(f) = \frac{\|\nabla f\|_{L^2}^2 \|f\|_{L^2}^2}{\|f\|_{L^4}^4}$$

among all functions $f \in H^1(\mathbb{R}^2)$, is found to be $C_{opt} = 2/\|Q\|_{L^2}^2$ [22].

Thus, the asymptotic object v satisfies

$$(2.15) \qquad \|v\|_{L^2} \geq \|Q\|_{L^2}.$$

d. Scaling back to the original variables.

Now we complete the proof. To prove (1.4), it suffices to show for any $\epsilon > 0$ that

$$\lim_{n\to\infty} \|u(t_n)\|_{L^2_{\{|x|<(T^*-t_n)^{s/2}\gamma(T^*-t_n)\}}} > \|Q\|_{L^2} - \epsilon.$$

Fix $\epsilon > 0$. Since $N(t_n)$ goes to ∞, we have

$$\lim_{n\to\infty} \|u(t_n)\|_{L^2_{\{|x|<(T^*-t_n)^{s/2}\gamma(T^*-t_n)\}}} = \lim_{n\to\infty} \|I_{N(t_n)}u(t_n)\|_{L^2_{\{|x|<(T^*-t_n)^{s/2}\gamma(T^*-t_n)\}}}.$$

Recalling the definition of the functions v_n, we have for all n

$$\|I_{N(t_n)}u(t_n)\|_{L^2_{\{|x|<(T^*-t_n)^{s/2}\gamma(T^*-t_n)\}}} = \|v_n\|_{L^2_{\{|y|<(T^*-t_n)^{s/2}\gamma(T^*-t_n)\sigma_n\}}}.$$

By (2.15) we know there exists $\rho > 0$ such that

$$\|v\|_{L^2_{\{|y|<\rho\}}} > \|Q\|_{L^2} - \epsilon.$$

Since, by Corollary 3.6, the σ_n go to ∞ at least as fast as $(T^* - t_n)^{-s/2}$, eventually $\rho < (T^* - t_n)^{s/2}\gamma(T^* - t_n)\sigma_n$, where $\gamma(z)$ is a positive function satisfying $\gamma(z) \uparrow \infty$ as $z \downarrow 0$. Thus,

$$\begin{aligned}
\lim_{n\to\infty} \|u(t_n)\|_{L^2_{\{|x|<(T^*-t_n)^{s/2}\gamma(T^*-t_n)\}}} &= \lim_{n\to\infty} \|v_n\|_{L^2_{\{|y|<(T^*-t_n)^{s/2}\gamma(T^*-t_n)\sigma_n\}}} \\
&\geq \lim_{n\to\infty} \|v_n\|_{L^2_{\{|y|<\rho\}}} \\
&\geq \|v\|_{L^2_{\{|y|<\rho\}}} \\
&> \|Q\|_{L^2} - \epsilon.
\end{aligned}$$

This concludes the proof of Theorem 1.1 under the assumptions that Proposition 2.1 and Corollary 3.6 hold true. $\qquad \square$

3. LOCAL-IN-TIME THEORY

In this section, we adapt the arguments in [6] to prove an almost conservation property for $E[I_N u]$ which plays a central role in the proof of Proposition 2.1. We begin by revisiting the Strichartz estimates and the classical proof [4] of local well-posedness of (1.1) for $u_0 \in H^s$, $s > 0$. We then explain a modification of the H^s local well-posedness result in which the H^s-norm of u_0 is replaced by $\|I_N\langle D\rangle u_0\|_{L^2}$. The modified local well-posedness result provides the space-time control used to prove the almost conservation of the modified energy $E[I_N u]$.

3.1. **Strichartz estimates.** We recall the classical Strichartz estimates [11] for the Schrödinger group $e^{it\Delta}$ on $\mathbb{R}_t \times \mathbb{R}_x^2$ (see also [12] for a unified presentation).

The ordered exponent pairs (q,r) are *admissible* if $\frac{2}{q} + \frac{2}{r} = 1$, $2 < q$. Note that $(3,6)$ and $(\infty, 2)$ are admissible. We define the *Strichartz norm* of functions $u : [0,T] \times \mathbb{R}^2 \to \mathbb{C}$,

$$(3.1) \qquad \|u\|_{S_T^0} = \sup_{(q,r) \text{ admissible}} \|u\|_{L_{t\in[0,T]}^q L_{x\in\mathbb{R}^2}^r}.$$

We will use the shorthand notation L_T^q to denote $L_{t\in[0,T]}^q$ and L_x^p for $L^p(\mathbb{R}_x^2)$. The Hölder dual exponent of q is denoted q', so $\frac{1}{q} + \frac{1}{q'} = 1$. The Strichartz estimates may be expressed as follows:

$$(3.2) \qquad \|u\|_{S_T^0} \le \|u(0)\|_{L^2} + \|(i\partial_t + \Delta)u\|_{L_T^{q'} L_{x\in\mathbb{R}^2}^{r'}}$$

where (q,r) is *any* admissible exponent pair.

The smoothing properties underlying our proof of the almost conservation of $E[I_N u]$ given below requires a careful control of the interaction between high and low frequency parts of the solution. Linear Strichartz estimates are not sufficient for this purpose. They are complimented by bilinear Strichartz estimates that were introduced in [1] and revisited in [10]. Let D^α denote the operator defined by $\widehat{D^\alpha u}(\xi) = |\xi|^\alpha \widehat{u}(\xi)$. Similarly, $\langle D \rangle^\alpha$ denotes the operator defined via $\widehat{\langle D \rangle^\alpha u}(\xi) = (1 + |\xi|^2)^{\frac{\alpha}{2}} \widehat{u}(\xi)$. We recall the following *a priori bilinear Strichartz estimate* [1] as expressed in Lemma 3.4 of [10]: For all $\delta > 0$ and any $u, v : [0,T] \times \mathbb{R}^2 \longmapsto \mathbb{C}$,

$$(3.3) \quad \|uv\|_{L_T^2 L_x^2} \le C(\delta) \quad \left(\|D^{-\frac{1}{2}+\delta} u(0)\|_{L_x^2} + \|D^{-\frac{1}{2}+\delta}(i\partial_t + \Delta)u\|_{L_T^1 L_x^2} \right)$$

$$\times \left(\|D^{\frac{1}{2}-\delta} v(0)\|_{L_x^2} + \|D^{\frac{1}{2}-\delta}(i\partial_t + \Delta)v\|_{L_T^1 L_x^2} \right).$$

Note that, in the inequality (3.3), a different amount of regularity is required for u and v. In the following analysis, we will use it with u, v being the projection of an H^s solution of (1.1) onto different frequency regimes.

3.2. **Standard H^s Local well-posedness.** We revisit the proof of local well-posedness of (1.1) for initial data in H^s, $s > 0$, extracting the features of the local-in-time theory required in the proof of the almost conservation property underlying our proof of Theorem 1.1 and Proposition 2.1.

Proposition 3.1 (H^s-LWP [4]). *For $u_0 \in H^s(\mathbb{R}^2)$, $s > 0$, the evolution $u_0 \longmapsto u(t)$ is well-posed on the time interval $[0, T_{lwp}]$ with*

$$(3.4) \qquad T_{lwp} = c_0 \|\langle D \rangle^s u_0\|_{L^2}^{-\frac{2}{s}},$$

for a constant c_0 and

$$(3.5) \qquad \|\langle D \rangle^s u\|_{S_{T_{lwp}}^0} \le 2\|\langle D \rangle^s u_0\|_{L_x^2}.$$

Proof. The initial value problem (1.1) is equivalent, by Duhamel's formula, to solving the integral equation

$$(3.6) \qquad u(t) = e^{it\Delta}u_0 + i\int_0^t e^{i(t-t')\Delta}(|u|^2 u(t'))dt'.$$

For a given function u, denote the right-hand side of (3.6) by $\Phi_{u_0}[u]$. We prove that $\Phi_{u_0}[\cdot]$ is a contraction mapping on the ball

$$(3.7) \qquad B_{L_T^3 L_x^6}(\rho) = \{u : [0,T] \times \mathbb{R}^2 \longmapsto \mathbb{C} \mid \|u\|_{L_T^3 L_x^6} < \rho\},$$

for sufficiently small ρ. In fact, we will show that $\|\Phi_{u_0}[u] - \Phi_{u_0}[v]\|_{S_T^0} < \frac{1}{2}\|u - v\|_{L_T^3 L_x^6}$ provided T is chosen small enough in terms of the H^s size of the initial data. We write $\Phi_{u_0}[u] - \Phi_{u_0}[v]$, observe the cancellation of the linear pieces, apply the S_T^0 norm, and use (3.2) to obtain

$$(3.8) \qquad \|\Phi_{u_0}[u] - \Phi_{u_0}[v]\|_{S_T^0} \le C\||u|^2 u - |v|^2 v\|_{L_T^1 L_x^2}.$$

(We have made the choice $q' = 1$ and $r' = 2$ in (3.2) to simplify the analysis below which emerges from the appearance of this norm in (3.3). This choice makes it convenient to perform the fixed point argument in $L_T^3 L_x^6$.)

By Hölder's inequality and simple algebra,

$$\||u|^2 u - |v|^2 v\|_{L_T^1 L_x^2} \le C(\|u\|_{L_T^3 L_x^6}^2 + \|v\|_{L_T^3 L_x^6}^2)\|u - v\|_{L_T^3 L_x^6} \le C\rho^2 \|u - v\|_{L_T^3 L_x^6}.$$

This proves $\Phi_{u_0}[\cdot]$ is indeed a contraction mapping on $B(\rho)$ if $\rho < \rho_0$, where ρ_0 is an explicit constant.

Thus, the sequence of Duhamel iterates $\{u^j\}$, defined by the recursion

$$(3.9) \qquad \begin{cases} u^0(t,x) = e^{it\Delta}u_0(x) \\ u^{j+1}(t) = e^{it\Delta}u_0 + i\int_0^t e^{i(t-t')\Delta}(|u^j|^2 u^j(t'))dt', \end{cases}$$

converges geometrically to the solution u provided that

$$(3.10) \qquad \|e^{it\Delta}u_0\|_{L_T^3 L_x^6} < \rho_0.$$

By Hölder's inequality in time and Sobolev's inequality in space,

$$(3.11) \qquad \|u\|_{L_T^3 L_x^6} \le CT^{\frac{1}{3}-\frac{1}{q}}\|\langle D\rangle^s u\|_{L_T^q L_x^r}$$

with $q > 3$ and $\frac{2}{r} = \frac{2}{6} + s$. We choose q so that (q,r) is admissible, *i.e.* $\frac{2}{q} + \frac{2}{r} = 1$. This gives $\frac{1}{3} - \frac{1}{q} = \frac{s}{2}$. Applying (3.11) and then (3.2), we have

$$(3.12) \qquad \begin{aligned} \|e^{it\Delta}u_0\|_{L_T^3 L_x^6} &\le T^{\frac{s}{2}}\|e^{it\Delta}\langle D\rangle^s u_0\|_{L_T^q L_x^r} \\ &\le T^{\frac{s}{2}}\|\langle D\rangle^s u_0\|_{L^2}. \end{aligned}$$

We choose $T = T_{lwp}$ so that the right-hand side of (3.12) is less than ρ_0, namely so that (3.4) holds. The Strichartz estimate implies that the zeroth iterate satisfies $\|u^0\|_{S_{T_{lwp}}^0} \le C\|u_0\|_{H^s}$. By the geometric convergence of the iterates, we have that

$$(3.13) \qquad \|u\|_{S_{T_{lwp}}^0} \le C\|u_0\|_{H^s}$$

and

$$\text{(3.14)} \qquad \|u\|_{L^3_{T_{lwp}} L^6_x} \le C\rho_0.$$

A posteriori, we can revisit the estimate for the solution of (3.6) and use the Leibnitz rule for fractional derivatives to prove the persistence of regularity property

$$\text{(3.15)} \qquad \|\langle D\rangle^s u\|_{S^0_T} \le 2\|\langle D\rangle^s u_0\|_{L^2}.$$

This follows from the Strichartz estimate on the linear term in (3.6) and the estimate

$$\text{(3.16)} \qquad \|\langle D\rangle^s(|u|^2 u)\|_{L^1_T L^2_x} \le C\|u\|^2_{L^3_T L^6_x} \|\langle D\rangle^s u\|_{L^3_T L^6_x} \le C\rho_0^2 \|\langle D\rangle^s u\|_{S^0_T}$$

where, in the last step, we used (3.14). The first inequality in (3.16) follows, for example, from Proposition 1.1 on page 105 of [19]. $\qquad\square$

Corollary 3.2 ([4]). *If $H^s \ni u_0 \longmapsto u(t)$ with $s > 0$ solves (1.1) for all t near enough to T^* in the maximal finite interval of existence $[0, T^*)$ then*

$$\text{(3.17)} \qquad C(T^* - t)^{-\frac{s}{2}} \le \|\langle D\rangle^s u(t)\|_{L^2_x}.$$

Note that the estimate (3.16) implies that, when we restrict attention to solutions of (1.1) with $t \in [0, T_{lwp}]$, we can essentially ignore the contributions involving the inhomogeneous terms appearing on the right side of (3.3) since these terms are controlled by the corresponding homogeneous terms. As a consequence of this and the bilinear Strichartz estimate, we prove the following lemma. We define $\mathbb{P}_{N_j}$ to be the Littlewood-Paley projection operator to frequencies of size $N_j \in 2^{\mathbb{N}}$, i.e. $\widehat{\mathbb{P}_{N_j} f}(\xi) = \chi_{\{\frac{1}{2} N_j < \xi < 2N_j\}} \hat{f}(\xi)$.

Lemma 3.3. *Suppose u solves (1.1) on the time interval $[0, T_{lwp}]$. Let $u_j = \mathbb{P}_{N_j} u$, for $j = 1, 2$ with $N_1 > N_2$. Then*

$$\text{(3.18)} \qquad \|u_1 u_2\|_{L^2_{T_{lwp}} L^2_x} \le C \left(\frac{N_2}{N_1}\right)^{\frac{1}{2}^-} \|u\|^2_{S^0_{T_{lwp}}},$$

The estimate (3.18) is also valid if u_j is replaced by $\overline{u_j}$.

This bilinear smoothing property of solutions of (1.1) is the key device underpinning the proof of the almost conservation of the modified energy in Proposition 3.7.

3.3. Modified H^s Local Well-posedness.

Proposition 3.4 (Modified H^s-LWP). *For $u_0 \in H^s$, $s > 0$, the (1.1) evolution $u_0 \longmapsto u(t)$ is well-posed on the time interval $[0, \widetilde{T}_{lwp}]$ with*

$$\text{(3.19)} \qquad \widetilde{T}_{lwp} = c_0 \|I_N \nabla u_0\|_{L^2_x}^{-\frac{2}{s}},$$

$$(3.20) \qquad \|I_N \langle D \rangle u\|_{S^0_{\widetilde{T}_{lwp}}} \leq 2 \|I_N \langle D \rangle u_0\|_{L^2_x}.$$

Proof. The proof is a modification of the arguments used to prove Proposition 3.1. Note first that the estimates (3.12) and (2.3) may be combined to give

$$(3.21) \qquad \|e^{it\Delta} u_0\|_{L^3_T L^6_x} \leq C T^{\frac{s}{2}} \|I_N \langle D \rangle u_0\|_{L^2}.$$

Thus, the previous analysis produces a solution u of (1.1) satisfying (3.14) and

$$(3.22) \qquad \|u\|_{S^0_{\widetilde{T}_{lwp}}} \leq C \|I_N \langle D \rangle u_0\|_{L^2}$$

provided that we choose $\widetilde{T}_{lwp}$ as in (3.19).

Next, we turn our attention toward the space-time regularity estimate (3.20). Since $e^{it\Delta}$ does not affect the magnitude of Fourier coefficients, (3.20) is clearly valid for the linear term in (3.6). Since the spaces appearing on both sides of the trilinear estimate (3.16) are translation invariant and (the first inequality in) (3.16) is valid, the modified estimate (3.20) follows directly from the interpolation lemma (Lemma 12.1 on page 108) in [5]. □

Remark 3.5. *A modification of* (3.18) *follows using the spacetime control* (3.20): *For $N_1 \geq N_2$ and for solutions u of* (1.1),

$$(3.23) \qquad \|I \langle D \rangle u_{N_1} I \langle D \rangle u_{N_2}\|_{L^2_{\widetilde{T}} L^2_x} \leq C \left(\frac{N_2}{N_1} \right)^{\frac{1}{2}-} \|I \langle D \rangle u\|^2_{S^0_{\widetilde{T}}}.$$

Corollary 3.6. *If $H^s \ni u_0 \longmapsto u(t)$ with $s > 0$ solves* (1.1) *for all t near enough to T^* in the maximal finite time interval of existence $[0, T^*)$*

$$(3.24) \qquad C(T^* - t)^{-\frac{s}{2}} \leq \|I_N \langle D \rangle u(t)\|_{L^2_x}.$$

Since we are studying here finite time blowup solutions, we will sometimes implicitly assume that $\|I_N \nabla u(t)\|_{L^2} > 1$.

3.4. Almost conservation law for the modified energy.

Proposition 3.7. *If $H^s \ni u_0 \longmapsto u(t)$ with $s > 0$ solves* (1.1) *for all $t \in [0, \widetilde{T}_{lwp}]$ then*
$$(3.25)$$
$$\sup_{t \in [0, \widetilde{T}_{lwp}]} |E[I_N u(t)]| \leq |E[I_N u(0)]| + C N^{-\alpha_4} \|I_N \langle D \rangle u(0)\|^4_{L^2_x} + C N^{-\alpha_6} \|I_N \langle D \rangle u(0)\|^6_{L^2_x},$$

with $\alpha_4 = \frac{3}{2}-$ and $\alpha_6 = 2-$.

Proof. We adapt arguments from [6] in which a similar result is proved using local well-posedness theory in the weighted $X_{s,b}$ spaces. We recall that the parameter N refers to the operator I_N defined in (2.2). In light of (3.20), it suffices to control the *energy increment* $|E[I_N u(t)] - E[I_N u(0)]|$ for

$t \in [0, \widetilde{T}_{lwp}]$ in terms of $\|I_N \langle D \rangle u\|_{S^0_{\widetilde{T}_{lwp}}}$. We define the set $*_n = \{(\xi_1, \ldots, \xi_n) : \Sigma \xi_i = 0\}$. An explicit calculation of $\partial_t E[I_N u]$ (carried out in detail in Section 3 of [6]) reveals that $|E[I_N u(t)] - E[I_N u(0)]|$ is controlled by the sum of the two space-time integrals:

$$(3.26) \quad E_1 = \left| \int_0^t \int_{*_4} [1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)}] \widehat{\Delta I \overline{u}}(\xi_1) \widehat{Iu}(\xi_2) \widehat{I \overline{u}}(\xi_3) \widehat{Iu}(\xi_4) \right|,$$

and
(3.27)

$$E_2 = \left| \int_0^t \int_{\tilde{\xi}_1 + \xi_4 + \xi_5 + \xi_6 = 0} [1 - \frac{m(\tilde{\xi}_1)}{m(\xi_4)m(\xi_5)m(\xi_6)}] \widehat{I(|u|^2 \overline{u})}(\tilde{\xi}_1) \widehat{Iu}(\xi_4) \widehat{I \overline{u}}(\xi_5) \widehat{Iu}(\xi_6) \right|.$$

We estimate the 4-linear expression (3.26) first. Let u_{N_j} denote $\mathbb{P}_{N_j} u$. When ξ_j is dyadically localized to $\{|\xi| \sim N_j\}$ we will write m_j to denote $m(\xi_j)$.

Lemma 3.8. *If* $H^s \ni u_0 \longmapsto u(t)$ *with* $s > 0$ *solves* (1.1) *for all* $t \in [0, \widetilde{T}_{lwp}]$ *then*

(3.28)
$$\left| \int_0^t \int_{*_4} [1 - \frac{m_1}{m_2 m_3 m_4}] \left(\widehat{\Delta I \overline{u}_{N_1}}(\xi_1) \widehat{Iu_{N_2}}(\xi_2) \widehat{I \overline{u}_{N_3}}(\xi_3) \widehat{Iu_{N_4}}(\xi_4) \right) \right|$$
$$\leq N^{-\alpha_4} \|I \langle D \rangle u\|^4_{S^0_{\widetilde{T}_{lwp}}} \prod_{j=1}^4 N_j^{0-},$$

with $\alpha_4 = \frac{3}{2}-$,

Proof. The analysis which follows will not rely upon the complex conjugate structure in the left-side of (3.28). Thus, there is symmetry under the interchange of the indices 2,3,4. We may therefore assume that $N_2 \geq N_3 \geq N_4$.

Case 1. $N \gg N_2$. On the convolution hypersurface $*_4$, this forces $N_1 \ll N$ as well, so the multiplier $[1 - \frac{m_1}{m_2 m_3 m_4}] = 0$ and the expression to be bounded vanishes.

Case 2. $N_2 \gtrsim N \gg N_3 \geq N_4$. This forces $N_1 \sim N_2$ on $*_4$. By the mean value theorem and simple algebra
(3.29)
$$\left| [1 - \frac{m_1}{m_2 m_3 m_4}] \right| = \left| [\frac{m(\xi_2) - m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)}] \right| \leq \left| \frac{\nabla m(\xi_2) \cdot (\xi_3 + \xi_4)}{m(\xi_2)} \right| \lesssim \frac{N_3}{N_2}.$$

We use the frequency localization of the u_{N_j} to renormalize the derivatives and multipliers to arrange for the appearance of $I \langle D \rangle u_{N_j}$. In the case under consideration, using (3.29) and the frequency localizations, the left-side of

(3.28) is controlled by

$$\frac{N_3}{N_2}N_1(N_2\langle N_3\rangle\langle N_4\rangle)^{-1}\left|\int_0^t\int_{*_4}\prod_{j=1}^4 I\langle D\rangle u_{N_j}(\xi_j)\right|.$$

We apply Cauchy-Schwarz to obtain the bound

$$\frac{N_3}{N_2}N_1(N_2\langle N_3\rangle\langle N_4\rangle)^{-1}\|I\langle D\rangle u_{N_1}I\langle D\rangle u_{N_3}\|_{L^2_{\tilde{T}_{lwp}}L^2_x}\|I\langle D\rangle u_{N_2}I\langle D\rangle u_{N_4}\|_{L^2_{\tilde{T}_{lwp}}L^2_x}.$$

By (3.23), we control by

$$\frac{N_3}{N_2}N_1(N_2\langle N_3\rangle\langle N_4\rangle)^{-1}\left(\frac{N_3N_4}{N_1N_2}\right)^{\frac{1}{2}-}\|I\langle D\rangle u\|^4_{S^0_{\tilde{T}_{lwp}}}$$

which simplifies to give the bound

$$(3.30)\qquad N^{-\frac{3}{2}+}(N_1N_2\langle N_3\rangle\langle N_4\rangle)^{0-}\|I\langle D\rangle u\|^4_{S^0_{\tilde{T}_{lwp}}}.$$

Case 3. $N_2\geq N_3\gtrsim N$. In this case, we use the trivial multiplier bound

$$(3.31)\qquad \left|[1-\frac{m_1}{m_2m_3m_4}]\right|\leq\frac{m_1}{m_2m_3m_4}.$$

Again, we pull the multiplier out and estimate the remaining integral using Lemma 3.3.

Case 3a. $N_1\sim N_2\geq N_3\gtrsim N$. We bound E_1 here by renormalizing the derivatives and multiplier, then pairing $u_{N_1}u_{N_3}$ and $u_{N_2}u_{N_4}$ and using Lemma (3.3) again:

$$\frac{m_1}{m_2m_3m_4}\left(\frac{N_3N_4}{N_1N_2}\right)^{\frac{1}{2}-}N_1(N_2N_3\langle N_4\rangle)^{-1}\|I\langle D\rangle u\|^4_{S^0_{\tilde{T}_{lwp}}}.$$

We reexpress this bound as

$$\frac{m_1}{m_2N_2^{\frac{1}{2}}m_3N_3^{\frac{1}{2}}m_4\langle N_4\rangle^{\frac{1}{2}}N_1^{\frac{1}{2}}}\|I\langle D\rangle u\|^4_{S^0_{\tilde{T}_{lwp}}}.$$

Since $m(x)$ is bounded from above by 1 and $m(x)\langle x\rangle^{\frac{1}{2}}$ is nondecreasing and bounded from below by 1, this is bounded by

$$(3.32)\qquad \frac{1}{NN_1^{\frac{1}{2}}}\|I\langle D\rangle u\|^4_{S^0_{\tilde{T}_{lwp}}}\leq N^{-\frac{3}{2}+}(N_1N_2N_3\langle N_4\rangle)^{0-}\|I\langle D\rangle u\|^4_{S^0_{\tilde{T}_{lwp}}}.$$

Case 3b. $N_2\sim N_3\geq N$. A similar analysis leads to the bound

$$\frac{m_1}{m_2m_3m_4}\left(\frac{N_1N_4}{N_3N_2}\right)^{\frac{1}{2}}N_1(N_2N_3\langle N_4\rangle)^{-1}\|I\langle D\rangle u\|^4_{S^0_{\tilde{T}_{lwp}}}$$

$$\leq\frac{1}{N^{\frac{1}{2}}N_2}\|I\langle D\rangle u_{N_j}\|^4_{S^0_{\tilde{T}_{lwp}}}$$

$$\leq N^{-\frac{3}{2}+}(N_1N_2N_3\langle N_4\rangle)^{0-}\|I\langle D\rangle u_{N_j}\|^4_{S^0_{\tilde{T}_{lwp}}}.$$

$\square$

A related case-by-case analysis combined with a trilinear estimate establishes the required 6-linear estimate for (3.27). We write m_{123} to denote $m(\xi_1 + \xi_2 + \xi+_3)$ and use N_{123} to denote the (dyadic) size of $\xi_1 + \xi_2 + \xi_3$. We will also use the similarly defined notation m_{456}. Note that N_{123} could be much smaller than $N_1, N_2,$ or N_3.

Lemma 3.9. *If* $H^s \ni u_0 \longmapsto u(t)$ *with* $s > 0$ *solves* (1.1) *for all* $t \in [0, \widetilde{T}_{lwp}]$ *then*

$$\int_0^t \int_{*_6} \left[1 - \frac{m_{123}}{m_4 m_5 m_6} \right] m_{123} \left(\widehat{\overline{u}}_{N_1}(\xi_1) \widehat{u}_{N_2}(\xi_2) \widehat{\overline{u}}_{N_3}(\xi_3) \right) \widehat{Iu}_{N_4}(\xi_4) \widehat{I\overline{u}}_{N_5}(\xi_5) \widehat{Iu}_{N_6}(\xi_6)$$

$$(3.33) \qquad\qquad\qquad \leq N^{-\alpha_6} \|I\langle D\rangle u\|^6_{S^0_{\widetilde{T}_{lwp}}} \prod_{j=1}^6 N_j^{0-}$$

with $\alpha_6 = 2-$.

Proof. We carry out a case-by-case analysis. By symmetry (since we will not use the complex conjugate structure), we may assume $N_4 \geq N_5 \geq N_6$.

Case 1. $N \gg N_4$. On $*_6$, this forces $N_{123} \sim N_4$ so the multiplier $\left[1 - \frac{m_{123}}{m_4 m_5 m_6} \right]$ vanishes.

Case 2. $N_4 \gtrsim N \geq N_5$. On $*_6$, $N_{123} \sim N_4$ in this case. By the mean value theorem,

$$\left| \left[1 - \frac{m_{123}}{m_4 m_5 m_6} \right] \right| = \left| \frac{m_4 - m_{456}}{m_4} \right| \leq \frac{N_5}{N_4}.$$

Applying this multiplier bound and the Cauchy-Schwarz inequality to the integral in (3.33) gives the bound

$$(N_4 \langle N_5 \rangle)^{-1} \frac{N_5}{N_4} \|\mathbb{P}_{N_{123}} I(\overline{u}_{N_1} u_{N_2} \overline{u}_{N_3}) Iu_{N_6}\|_{L^2_{Tx}} \|I\langle D\rangle u_{N_4} I\langle D\rangle u_{N_5}\|_{L^2_{Tx}}.$$

By Hölder's inequality and Lemma 3.3, we control the above expression by
(3.34)

$$(N_4 \langle N_5 \rangle)^{-1} \frac{N_5}{N_4} \|\mathbb{P}_{N_{123}} I(\overline{u}_{N_1} u_{N_2} \overline{u}_{N_3})\|_{L^2_{Tx}} \|Iu_{N_6}\|_{L^\infty_{Tx}} \left(\frac{N_5}{N_4} \right)^{\frac{1}{2}} \|I\langle D\rangle u\|_{S^0_{\widetilde{T}_{lwp}}} \|I\langle D\rangle u\|_{S^0_{\widetilde{T}_{lwp}}}.$$

By Sobolev's inequality on functions with frequency support localized to a dyadic shell on R_x^2,

$$(3.35) \qquad\qquad \|Iu_{N_6}\|_{L^\infty_{Tx}} \leq \|I\langle D\rangle u_{N_6}\|_{L^\infty_T L^2_x}.$$

In order to continue the proof of Lemma 3.9, we need an estimate of the term $\|\mathbb{P}_{N_{123}} I(\overline{u}_{N_1} u_{N_2} \overline{u}_{N_3})\|_{L^2_{Tx}}$. This is the purpose of the next lemma. Let $N_1^\dagger \geq N_2^\dagger \geq N_3^\dagger$ denote the decreasing rearrangement of N_1, N_2, N_3.

Lemma 3.10.

$$(3.36) \qquad \|\mathbb{P}_{N_{123}} I(\overline{u}_{N_1} u_{N_2} \overline{u}_{N_3})\|_{L^2_{Tx}} \leq \langle N_1^\dagger \rangle^{-\frac{1}{2}} \prod_{j=1}^{3} \|I\langle D \rangle u_{N_j}\|_{S^0_{\tilde{T}_{lwp}}} .$$

Proof. Again, we will not use the complex conjugate structure so we may assume that $N_1 \geq N_2 \geq N_3$. The projection $\mathbb{P}_{N_{123}}$ allows us to control the left-hand side of (3.36) by

$$m_{123}\|\overline{u}_{N_1} u_{N_2} \overline{u}_{N_3}\|_{L^2_{Tx}} \leq m_{123}\|u_{N_1}\|_{L^4_{Tx}}\|u_{N_2}\|_{L^4_{Tx}}\|u_{N_3}\|_{L^\infty_{Tx}}$$

where we used Hölder's inequality.

We renormalize the derivatives and use a (dyadically localized) Sobolev inequality as in (3.35) to get the bound

$$\frac{m_{123}}{m_1 m_2 m_3}\langle N_1 \rangle^{-1} \langle N_2 \rangle^{-1}\|I\langle D \rangle u_{N_1}\|_{L^4_{Tx}}\|I\langle D \rangle u_{N_2}\|_{L^4_{Tx}}\|I\langle D \rangle u_{N_3}\|_{L^\infty_T L^2_x}.$$

Since the norms appearing in the above expression are admissible we focus our attention upon the prefactor and bound with the expression

$$\frac{m_{123} N_3^{\frac{1}{2}}}{\langle N_1 \rangle^{\frac{1}{2}} \langle N_2 \rangle^{\frac{1}{2}} m_1 \langle N_1 \rangle^{\frac{1}{2}} m_2 \langle N_2 \rangle^{\frac{1}{2}} m_3 \langle N_3 \rangle^{\frac{1}{2}}} \prod_{j=1}^{3} \|I\langle D \rangle u_{N_j}\|_{S^0_{\tilde{T}_{lwp}}} .$$

Since $m(x) \leq 1$, $m(x)\langle x \rangle^{\frac{1}{2}}$ is nondecreasing, and $N_3 \leq N_2$, this proves (3.36). $\qquad \square$

We use (3.36) and (3.35) on (3.34) to complete the Case 2 analysis. The left-hand side of (3.33) is bounded by

$$(3.37) \qquad (N_4 \langle N_5 \rangle)^{-1} \left(\frac{N_5}{N_4}\right)^{\frac{3}{2}} \langle N_1^\dagger \rangle^{-\frac{1}{2}} \|I\langle D \rangle u\|^6_{S^0_{\tilde{T}_{lwp}}}$$

$$(3.38) \qquad \leq \frac{N_5^{\frac{1}{2}}}{N_4^{\frac{5}{2}}}(N_1^\dagger)^{-\frac{1}{2}} \|I\langle D \rangle u\|^6_{S^0_{\tilde{T}_{lwp}}}$$

$$(3.39) \qquad \leq N^{-2+}(N_1 \ldots \langle N_5 \rangle \langle N_6 \rangle)^{0-} \prod_{j=1}^{6} \|I\langle D \rangle u\|^6_{S^0_{\tilde{T}_{lwp}}} .$$

Case 3. $N_4 \geq N_5 \geq N$. We will use the trivial multiplier estimate

$$\left|\left[1 - \frac{m_{123}}{m_4 m_5 m_6}\right]\right| \leq \frac{m_{123}}{m_4 m_5 m_6}.$$

Familiar steps lead to the bound

$$\frac{m_{123}}{m_4 m_5 m_6}\langle N_1^\dagger \rangle^{-\frac{1}{2}} \left(\frac{N_5}{N_4}\right)^{\frac{1}{2}} (N_4 N_5)^{-1} \|I\langle D \rangle u\|^6_{S^0_{\tilde{T}_{lwp}}}$$

which we recast as

$$(3.40) \qquad \frac{m_{123}}{m_4 N_4^{\frac{1}{2}} m_5 N_5^{\frac{1}{2}} m_6 \langle N_1^\dagger \rangle^{\frac{1}{2}} N_4} \|I\langle D \rangle u\|^6_{S^0_{\tilde{T}_{lwp}}} .$$

Case 3a. $N_6 \geq N$. We express the prefactor in (3.40) as

$$\frac{m_{123} N_6^{\frac{1}{2}}}{m_4 N_4^{\frac{1}{2}} m_5 N_5^{\frac{1}{2}} m_6 N_6^{\frac{1}{2}} \langle N_1^{\dagger} \rangle^{\frac{1}{2}} N_4} \leq \frac{1}{N^{\frac{3}{2}} N_4^{\frac{1}{2}}} \leq N^{-2+}(N_1 \ldots N_6)^{0-}.$$

Case 3b. $N_6 \leq N$. Here we have $m_6 = 1$ so we can bound the prefactor in (3.40) by

$$\frac{1}{N N_4} \leq N^{-2+}(N_1 N_2 N_3 N_4)^{0-}.$$

$\square$

Since (3.26) and (3.27) control the increment in the modified energy, it suffices to prove appropriate bounds on these integrals to obtain (3.25). Lemmas (3.8), (3.9) provide estimates for dyadically localized contributions to the integrals (3.26), (3.27), respectively. Since the estimates (3.28) and (3.33) have the helpful decay factors N_j^{0-}, we can sum up the dyadic contributions, and apply Cauchy-Schwarz to complete the proof of (3.25).

$\square$

4. Modified kinetic energy dominates modified total energy

In this section, we prove Proposition 2.1.

Proof of Proposition 2.1. When $s \geq 1$, we set $I_{N(T)} = Identity$ by choosing $N(T) = +\infty$. Since the (unmodified) energy is conserved while the kinetic energy blows up as time approaches T^*, the estimate (2.8) is obvious with $p(s) = 0$. We therefore restrict attention to $s \in (s_Q, 1)$, with s_Q to be determined at the end of the proof.

Fix $s \in (s_Q, 1)$ and T near T^*. Let $N = N(T)$ (to be chosen). Set $\delta = c_0 (\Sigma(T))^{-\frac{2}{s}} > 0$ with c_0 the small fixed constant in (3.19). Note that δ is the time of local well-posedness guaranteed by Proposition 3.4 for initial data of size $\Sigma(T)$, which is the largest value that the modified kinetic energy achieves up to time T. Thus the interval $[0, T]$ may be partitioned into $J = C \frac{T}{\delta}$ δ-sized intervals on which the modified local well-posedness result uniformly applies. More precisely, $[0, T] = \bigcup_{j=1}^{J} I_j$, $I_j = [t_j, t_{j+1})$, $t_0 = 0$, $t_{j+1} = t_j + c\delta$, and we have at each t_j,

$$\|I_N \langle D \rangle u(t_j)\|_{L^2} = \sigma(t_j) \leq \Sigma(T).$$

In addition, δ has been taken sufficiently small so that we can apply the almost conservation law Proposition 3.7 on each of the I_j. We now accumulate increments to the energy and have that

$$|E[I_N u(T)]| \leq |E[I_N u(0)]| + C \frac{T^*}{\delta} [N^{-\alpha_4} \Sigma^4(T) + N^{-\alpha_6} \Sigma^6(T)]$$

$$\leq N^{2(1-s)} \lambda(0) + C \frac{T^*}{\delta} N^{-\alpha_4} \Sigma^4(T) + C \frac{T^*}{\delta} N^{-\alpha_6} \Sigma^6(T).$$

By the choice of δ we see, dismissing irrelevant constants, that

$$|E[I_N u(T)]| \lesssim N^{2(1-s)} + N^{-\alpha_4} \Sigma^{4+\frac{2}{s}}(T) + N^{-\alpha_6} \Sigma^{6+\frac{2}{s}}(T).$$

Using (2.3), we can switch from Σ to Λ:
(4.1)
$$|E[I_N u(T)]| \lesssim N^{2(1-s)} + N^{-\alpha_4 + (4+\frac{2}{s})(1-s)} \Lambda^{4+\frac{2}{s}}(T) + N^{-\alpha_6 + (6+\frac{2}{s})(1-s)} \Lambda^{6+\frac{2}{s}}(T).$$

We choose $N = N(\Lambda)$ so that the first and third terms in (4.1) give comparable contributions. A calculation reveals that the second term in (4.1) produces a smaller correction. Thus, with the choice

$$(4.2) \qquad\qquad N = \Lambda^{\frac{6+\frac{2}{s}}{\alpha_6 - (4+\frac{2}{s})(1-s)}},$$

the Proposition 2.1 is established with

$$(4.3) \qquad\qquad p(s) = \frac{6 + \frac{2}{s}}{\alpha_6 - (4 + \frac{2}{s})(1-s)} 2(1-s).$$

Note that $p(s) < 2$ reduces to to a quadratic condition on s. Specifically

$$10s^2 + (\alpha_6 - 6)s - 4 > 0.$$

For $\alpha_6 = 2-$ this yields

$$s > s_Q = \frac{1}{5} + \frac{1}{5}\sqrt{11} \sim 0.863.$$

$\square$

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